Int. J. Solids Structures, 1968, Vol. 4, pp. 469 to 475. Pergamon Press. Printed in Great Britain

OPTIMAL DESIGN OF MULTI-PURPOSE STRUCTURES*

WILLIAM PRAGER

Department of the Aerospace and Mechanical Engineering Sciences
University of California, San Diego
La Jolla, California

and

R. T. SHIELD

Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California

Abstract—Minimum-weight design of a sandwich member that has to serve as a tie in some circumstances and as a beam in others is used to illustrate a general method of optimal design of sandwich structures that have to meet several design requirements. The extension of the method to solid construction is discussed.

1. INTRODUCTION

A RECENT paper [1] presented a uniform method of treating a variety of problems of minimum-weight design of sandwich structures that have to meet a single design requirement. Optimal elastic design for maximum stiffness, maximum fundamental frequency, or maximum buckling load, and optimum plastic design for maximum safety were treated as examples. In the present paper, the optimal elastic design of a beam-tie for given transversal and longitudinal stiffness is used to illustrate the extension of the method to two or more design requirements and to solid structures.

Consider a straight sandwich member of the length $2l$ that has to serve as a tie in some circumstances (Fig. 1a) and as a beam in others (Fig. 1b). The core of the member is to have constant height $2h$ and constant breadth b; the identical face sheets are to have constant breadth b but variable thickness $t(x)$, where x denotes distance measured along the member. All direct stresses are to be carried by the face sheets. The specific axial stiffness at the cross section x is therefore

$$s(x) = 2Ebt(x), \tag{1.1}$$

where E is Young's modulus, while the specific bending stiffness is $h^2s(x)$.

The variation of the thickness $t(x)$ is to be determined in such a manner that the member experiences an elongation not exceeding the given value 2λ when it is subjected to a longitudinal load L (Fig. 1a) and a maximum deflection not exceeding the given value δ when it is subjected to a transverse load $2T$ at the center of the span (Fig. 1b). Moreover, the face sheets are to have minimum weight. On account of the symmetry of loading and support

* This research was supported in part (W.P.) by the Advanced Research Projects Agency (Project DEFENDER) under Contract 31-124-ARO-D-257 monitored by the U.S. Army Research Office—Durham.

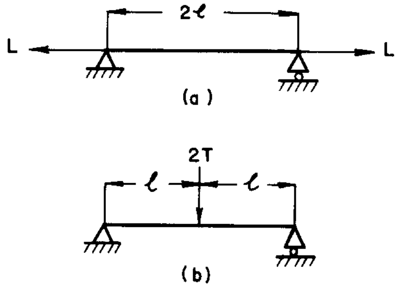

FIG. 1. Member acting as tie or beam.

with respect to the cross section $x = l$ (Fig. 1), we may set $t(x) = t(2l-x)$ and restrict the discussion to the part between the cross sections $x = 0$ and $x = l$. Since $t(x)$ is the only variable in the expression (1.1) for $s(x)$, minimizing the weight of the face sheets is equivalent to minimizing the integral $\int_0^l s(x)\,dx$.

2. TIE ACTION

When the member is acting as a tie, the longitudinal displacement $u(x)$ satisfies the following differential equation and boundary conditions

$$su' = L \quad \text{in} \quad 0 \le x \le l, \tag{2.1}$$

$$u(0) = 0, \qquad u(l) = \lambda. \tag{2.2}$$

If $s(x)$ is given, the two boundary conditions (2.2) may be imposed only if L in (2.1) is regarded as an unknown constant. The principle of minimum potential energy then states that the solution $u(x)$ of the boundary value problem (2.1), (2.2) yields a smaller value of the expression $\int_0^l su'^2\,dx$ than any other kinematically admissible displacement field, that is, any other continuous displacement field satisfying (2.2). Moreover, the minimum of $\int_0^l su'^2\,dx$ has the value $L\lambda$.

Let $\bar{s}(x)$ be an alternative design that experiences the same elongation under the same load, and denote its longitudinal displacement by $\bar{u}(x)$. Thus,

$$\lambda L = \int_0^l su'^2\,dx = \int_0^l \bar{s}\bar{u}'^2\,dx. \tag{2.3}$$

Since the displacement $u(x)$ is kinematically admissible for the design $\bar{s}(x)$, the principle of minimum potential energy applied to this design furnishes the inequality

$$\int_0^l \bar{s}\bar{u}'^2\,dx < \int_0^l \bar{s}u'^2\,dx. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\int_0^l (s-\bar{s})u'^2\,dx < 0. \tag{2.5}$$

If the longitudinal displacement $u(x)$ of the design $s(x)$ satisfies

$$\mu^2 u'^2 = 1, \tag{2.6}$$

where μ is a dimensionless constant, it follows from (2.5) that the design $s(x)$ is lighter than any other design $\bar{s}(x)$ that experiences the same elongation under the same load.

If the member is only to act as a tie, its minimum-weight design is obtained by determining μ from the differential equation (2.6) and the boundary conditions (2.2) and then substituting u' from (2.6) into (2.1) and solving for s. Thus,

$$s = s_L = \frac{l}{\lambda} L. \tag{2.7}$$

The optimal tie design has constant specific axial stiffness.

3. BEAM ACTION

When the member is acting as a beam under a central load $2T$, the transverse displacement $v(x)$ satisfies the following differential equation and boundary conditions

$$h^2 s v'' = -Tx \quad \text{in} \quad 0 \leq x \leq l, \tag{3.1}$$

$$v(0) = 0, \quad v'(l) = 0, \quad v(l) = \delta. \tag{3.2}$$

If $s(x)$ is given, the three boundary conditions (3.2) may be imposed only if T in (3.1) is regarded as an unknown constant. The principle of minimum potential energy then states that the solution $v(x)$ of the boundary value problem (3.1), (3.2) furnishes a smaller value of the expression $\int_0^l sh^2 v''^2 \, dx$ than any other kinematically admissible deflection, that is, any other continuously differentiable deflection satisfying (3.2). Moreover, the minimum value of $\int_0^l sh^2 v''^2 \, dx$ is $T\delta$.

If $\bar{s}(x)$ is an alternative design experiencing the same maximum deflection under the same load, the principle of minimum potential energy may be used in a similar way as in Section 2 to derive the inequality

$$\int_0^l (s - \bar{s}) h^2 v''^2 \, dx < 0. \tag{3.3}$$

This furnishes the optimality condition

$$\nu^2 h^2 v''^2 = 1, \tag{3.4}$$

where ν is a dimensionless constant, when the member is to act only as a beam. The corresponding optimal design is found to be

$$s = s_T = \frac{l^2 x}{2h^2 \delta} T \quad \text{for } 0 \leq x \leq l. \tag{3.5}$$

4. TIE ACTION AND BEAM ACTION

When both kinds of action must be considered, the inequalities (2.5) and (3.3) may be combined with positive multipliers to yield the optimality condition

$$\mu^2 u'^2 + \nu^2 h^2 v''^2 = 1, \tag{4.1}$$

where μ and ν are dimensionless constants.

The weight of the optimal design is proportional to $\int_0^l s\,\mathrm{d}x$. A simple expression for this integral can be obtained by multiplying (4.1) by s, using (2.1) and (3.1), and integrating the resulting equation between the limits $x = 0$ and $x = l$. Thus,

$$\int_0^l s\,\mathrm{d}x = \mu^2 L\lambda + \nu^2 T\delta. \tag{4.2}$$

If the transverse load $2T$ is sufficiently small, tie action alone may govern the optimal design even though in principle both kinds of action have to be considered. In this case, $\nu = 0$ in (4.1), and this optimality condition reduces to (2.6) yielding the design (2.7). Substituting this expression for s into (3.1), integrating under the first two conditions (2.2), and requiring that $v(l)$ must not exceed δ, we obtain

$$T \leq T_0 = 3\frac{\delta}{\lambda}\frac{h^2}{l^2}L, \tag{4.3}$$

and tie action alone governs the design when this inequality holds.

The possibility that beam action alone governs the optimal design may be explored in a similar manner. It is found that this case cannot occur unless $L = 0$. Accordingly, both tie action and beam action influence the optimal design whenever $T > T_0$ and $L \neq 0$. The manner in which the specific axial stiffness s of the optimal design then depends on x is found by multiplying (4.1) by s^2 and using (2.1) and (3.1). Thus,

$$s = (\mu^2 L^2 + \nu^2 T^2 x^2/h^2)^{1/2} = L(\mu^2 + \xi^2)^{1/2}, \tag{4.4}$$

where

$$\xi = \omega x/l, \qquad \omega = \nu l T/(hL). \tag{4.5}$$

Note that ξ is a dimensionless distance measured along the member.

Substituting (4.4) into (2.1) and integrating under the first condition (2.2), we find

$$u = (l/\omega)\sinh^{-1}(\xi/\mu). \tag{4.6}$$

With

$$\eta = \omega\lambda/l, \tag{4.7}$$

use of (4.6) and the second condition (2.2) furnishes

$$\mu = \eta l/(\lambda \sinh \eta). \tag{4.8}$$

Similarly, substitution of (4.4) into (3.1) and use of the first two conditions (3.2) yields

$$v = \frac{l^3 T}{2h^2\omega^3 L}\{2\xi(\mu^2+\omega^2)^{1/2} - \xi(\mu^2+\xi^2)^{1/2} - \mu^2\sinh^{-1}(\xi/\mu)\}. \tag{4.9}$$

The third condition (3.2) now furnishes

$$\frac{\sinh 2\eta - 2\eta}{\eta(\cosh 2\eta - 1)} = \frac{2\delta h^2 L}{\lambda l^2 T}. \tag{4.10}$$

When this transcendental equation is solved for η, the values of μ and ω are obtained from (4.8) and (4.7). Finally, the optimal design is determined from (4.4).

For example, the specifications

$$h/l = 1/20, \qquad \delta/l = 1/425, \qquad \lambda/l = 1/1000, \qquad T/L = 1/50 \tag{4.11}$$

used in (4.10) yield

$$\eta = 1{\cdot}00, \qquad \mu = 850, \qquad \omega = 1000, \tag{4.12}$$

and

$$s = L[850^2+(1000x/l)^2]^{1/2}. \tag{4.13}$$

As is readily verified from (4.2), $T/L = 0{\cdot}020 > T_0/L = 0{\cdot}0176$. The weight of the face sheets for the optimal design is 10% less than that for the design with constant thickness. Since $s = EA$, where A is the combined cross-sectional area of the cover sheets, the stress σ_L induced by the load L in these sheets is given by

$$\sigma_L = \frac{L}{A} = E[850^2+(1000x/l)^2]^{-1/2}. \tag{4.14}$$

This stress decreases from $E/850$ at $x = 0$ to about $E/1312$ at $x = l$. On the other hand, the absolute value of the bending stress σ_T induced by the transverse load $2T$ is found to be

$$\sigma_T = 0{\cdot}4E\frac{x}{l}[850^2+(1000x/l)^2]^{-1/2}. \tag{4.15}$$

It follows from (4.14) and (4.15) that

$$(850\sigma_L)^2+(2500\sigma_T)^2 = E^2. \tag{4.16}$$

This is the form assumed by the optimality condition (4.1) for the present example when the unit extension u' and the curvature $-v''$ are expressed in terms of the stresses σ_L and σ_T.

5. OTHER DESIGN REQUIREMENTS

The example discussed in Sections 2 through 4 involved two design requirements, which were both concerned with stiffness. The extension of the method to several design requirements involving other structural properties is immediate when suitable minimum principles are known for these other properties. Suppose, for instance, that in some circumstances the member considered above may also have to act as a column and that its Euler load is therefore required to have at least the given value P. The deflection $w(x)$ in elastic buckling satisfies the following differential equation and boundary conditions:

$$h^2sw'' = -Pw, \tag{5.1}$$

$$w(0) = w'(l) = 0, \tag{5.2}$$

and the Euler load may be found as the minimum value of a Rayleigh quotient. In analogy to (2.3) and (2.4), we now have

$$P = \frac{\int_0^l sh^2w''^2\,dx}{\int_0^l w'^2\,dx} = \frac{\int_0^l \bar{s}h^2\bar{w}''^2\,dx}{\int \bar{w}'^2\,dx} < \frac{\int_0^l \bar{s}h^2w''^2\,dx}{\int_0^l w'^2\,dx}. \tag{5.3}$$

which yields

$$\int (s-\bar{s})h^2w''^2 \, dx < 0. \tag{5.4}$$

If the given Euler load is the only design requirement to be considered, (5.4) furnishes the optimality condition

$$\rho^2h^2w''^2 = 1, \tag{5.5}$$

where ρ is a dimensionless constant. The corresponding optimal design has been discussed in [1]. If all three design requirements are relevant, however, the optimality condition takes the form

$$\lambda^2u'^2 + \nu^2h^2v''^2 + \rho^2h^2w''^2 = 1. \tag{5.6}$$

By taking one or two of the constants λ, ν, ρ in (5.6) as zero, we obtain the optimality conditions for the cases in which only two or one of the three design requirements are relevant.

It is worth noting that, as the first eigenfunction of the homogeneous boundary value problem (5.1), (5.2), the function $w(x)$ is only determined to within a constant factor. Accordingly, the factor ρ^2 in (5.6) may be omitted whenever the required Euler load affects the optimal design.

We shall not pursue this problem but discuss instead the application of the method to structures that are not of sandwich type. As has been pointed out in [1], a realistic problem of minimum-weight design involves restrictions on the size of the structural members. Consider, for instance, the minimum-weight design of the beam in Fig. 1b for a given deflection δ under the central load $2T$. To obtain a definite minimum-weight design without invoking considerations of lateral stability, we must restrict the space available for the beam, say to a rectangular prism of given height $2h$ and breadth b. The optimal design then is a sandwich beam that fully uses this available space and places the material in direct stress as close to the relevant faces of this space as is possible.

Occasionally, however, solid cross sections may have to be considered, because sandwich construction is not practical. As has been shown in [2] for the special case of optimal plastic design for given load-carrying capacity, the optimality condition for sandwich construction is readily adapted to solid construction, but the modified condition ensures a relative rather than absolute minimum of weight. Rather than discussing the modification in general terms, we shall briefly indicate the manner in which the treatment of Section 2 through 4 must be changed if the member is to have a solid rectangular cross section of constant breadth b and variable height $2H$.

The specific axial and bending stiffnesses s and h^2s in Sections 2 through 4 must then be replaced by $A = 2EbH$ and $B = 2EbH^3/3$, respectively. Instead of considering an arbitrary alternative design $\bar{s}$ meeting the same requirements, we now consider a neighboring design with the height $2(H+\Delta H)$, where $\Delta H/H \ll 1$. If higher powers of ΔH are neglected, the specific axial and bending stiffnesses of this design are $A+2Eb\Delta H$ and $B+2Eb\Delta H^2\Delta H$. It now follows from (2.5) and (3.3) that a design for which

$$\mu^2u'^2 + \nu^2H^2v''^2 = 1 \tag{5.7}$$

is lighter than any *neighboring* design meeting the same requirements. Note that H in (5.7) is a function of x, whereas h in (4.1) is a constant.

The optimality condition (5.7) may be used in a similar manner as (4.1) to obtain an expression for $\int_0^l 2EbH\,dx$, which is proportional to the weight of the optimal design. Thus,

$$\int_0^l 2EbH\,dx = \mu^2 L\lambda + 3\nu^2 T\delta. \tag{5.8}$$

Note, however, that the values of μ and ν in (5.8) need not be the same as those in (4.2).

REFERENCES

[1] W. PRAGER and J. E. TAYLOR, Problems of optimal structural design. *J. appl. Mech.* To be published.
[2] R. T. SHIELD, On the optimum design of shells. *J. appl. Mech.* **27**, 316-322 (1960).

(*Received* 14 *August* 1967)

Абстракт—Расчет на минимум веса сломстого элемента, работающего как тяга на некоторых расстояниях, и как балка на других, исполрзуется для иллюстрации общего метода оптимального расчета слоистых конструкций, которые должны удовлетворять некоторым требованиям расчета. Приводится применения этого метода к расчету монолитных конструкций.